

# ON THE REDUCED GRADES OF MODULES OVER COMMUTATIVE RINGS

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**ABSTRACT.** Let  $R$  be a commutative Noetherian ring. Recently, Dibaei and Sadeghi have studied the reduced grade of a horizontally linked  $R$ -module  $M$  of finite  $G_C$ -dimension, where  $C$  is a semidualizing  $R$ -module. In this paper, we highly refine their results. In particular, our main result removes the assumptions that  $M$  is horizontally linked and  $M$  has finite  $G_C$ -dimension.

## 1. INTRODUCTION

A purpose of this paper is to generalize some results of a paper of Dibaei and Sadeghi [4]. In [4] they proved many theorems for a horizontally linked module of finite  $G_C$ -dimension. However, the conditions “horizontally linked” and “finite  $G_C$ -dimension” are too strong for some theorems. Let  $R$  be a commutative Noetherian ring,  $M$  a finitely generated  $R$ -module, and  $C$  a semidualizing module. We say that  $M$  satisfies the condition  $\tilde{S}_i^C$  if  $\text{depth } M_p \geq \inf\{i, \text{depth } C_p\}$  for integer  $i$  and all  $p \in \text{Spec } R$ . Recall that  $M$  is called  $n$ - $C$ -torsionfree if  $\text{Ext}_R^i(\text{Tr}_C M, C) = 0$  for any  $1 \leq i \leq n$ . Let  $t^C(M)$  stand for the supremum of the integers  $n$  such that  $M$  is  $n$ - $C$ -torsionfree. The main result of this paper is the following.

**Theorem 1.1.** *If  $t^C(M) < \infty$ , then the following are equivalent.*

- (1) *For every integer  $i$ ,  $M$  is  $n$ - $C$ -torsionfree if and only if  $M$  satisfies  $\tilde{S}_i^C$ .*
- (2) *There exists an associated prime  $p$  of  $\text{Ext}_R^{t^C(M)+1}(\text{Tr}_C M, C)$  such that  $t^C(M) + 1 \leq \text{depth } C_p$ .*

Auslander and Bridger proved the following; see [1, Theorem 4.25].

- (a) If  $M$  is  $i$ - $R$ -torsionfree, then  $M$  satisfies  $\tilde{S}_i^R$  for an integer  $i$ .
- (b) If  $M$  has finite  $G$ -dimension, then  $M$  satisfies  $i$ - $R$ -torsionfree if and only if  $M$  satisfies  $\tilde{S}_i^R$  for every integer  $i$ .

Here, the converse of (b) is not true in general. For example, let  $(R, m, k)$  be a non-Gorenstein local ring with positive depth. Let  $M$  be the first syzygy of the  $R$ -module  $k$ . Then  $M$  is 1-torsionfree. By the depth lemma, we have  $\text{depth } M = 1$ . Therefore,  $M$  satisfies  $i$ - $R$ -torsionfree if and only if  $M$  satisfies  $\tilde{S}_i^R$  for every integer  $i$ . But  $M$  has infinite  $G$ -dimension.

Let  $\tilde{S}^C(M)$  denote the supremum of the integers  $i$  such that  $M$  satisfies  $\tilde{S}_i^C$ . Then Theorem 1.1(1) nothing but says that  $M$  satisfies  $t^C(M) = \tilde{S}^C(M)$ . By this we can generalize theorems of Dibaei and Sadeghi.

## 2. THE DEFINITIONS

Throughout this paper, let  $\Lambda, \Gamma$  be rings, and a module means a left (or right) module. Let  $M$  be  $\Gamma$ -module which have a resolution of finitely generated projective  $\Gamma$ -module,  $C$  be  $\Lambda$ - $\Gamma$ -bimodule, and  $n \geq 1$  be an integer. Let  $d$  be a function which detemind a nonnegative integer or  $\infty$  for every  $\Lambda$ -modules. In this paper we need many definitions.

We consider the following conditions for  $d$ . Let  $0 \rightarrow K \rightarrow L \rightarrow N \rightarrow 0$  be a short exact sequence of  $\Lambda$ -modules.

- ( $\widetilde{d1}$ ) if  $d(L) > d(K)$ , then  $d(K) \geq d(N) + 1$ ,
- ( $d1$ ) if  $d(L) > d(K)$ , then  $d(K) = d(N) + 1$ ,
- ( $d2$ ) if  $d(L) > d(N)$ , then  $d(K) = d(N) + 1$ ,
- ( $d3$ ) if  $d(L) < d(N)$ , then  $d(K) \leq d(L)$ .

We have  $(d1) \Rightarrow (\widetilde{d1})$ , and  $(\widetilde{d1})(d2) \Rightarrow (d1)$  easily and if  $d$  satisfies  $(\widetilde{d1})$  we have  $d(0) \geq \sup\{d(N) \mid R\text{-module } N\}$  because there exist a exact sequence  $0 \rightarrow 0 \rightarrow L \rightarrow L \rightarrow 0$  for any  $\Lambda$ -module  $L$ . For example let  $H^i$  be an  $i$ -th right derived functor of a left exact functor to an abelian category from the category of  $\Lambda$ -module. For a  $\Lambda$ -module  $N$ , we put  $d(N) = \inf\{i \mid H^i(N) \neq 0\}$ . Then  $d$  satisfies  $(d1)(d2)(d3)$  because there exist the long exact sequence for a short exact sequence.

The following Propositions follows immediately.

**Proposition 2.1.** *Let  $i$  be a nonnegative integer and  $0 \rightarrow K \rightarrow L^0 \rightarrow L^1 \rightarrow \dots \rightarrow L^i \rightarrow N \rightarrow 0$  be a exact sequence of  $\Lambda$ -module. Then following holds.*

- (1) *If  $d$  satisfies  $(\widetilde{d1})$  (resp.  $(d1)$ ) and  $d(K) - j < d(L^j)$  for  $0 \leq j \leq i$ , then  $d(K) \geq d(N) + i + 1$  (resp.  $d(K) = d(N) + i + 1$ ).*
- (2) *If  $d$  satisfies  $(d2)$  and  $d(N) + i - j < d(L^j)$  for  $0 \leq j \leq i$ , then  $d(K) = d(N) + i + 1$ .*

*Proof.* (1); Let  $k$  be a nonnegative integer. We assume that the assetion is true for  $i = k - 1$  and prove the case of  $i = k$ . We have exact sequences  $0 \rightarrow K \rightarrow L^0 \rightarrow L^1 \rightarrow \dots \rightarrow L^{k-1} \rightarrow T \rightarrow 0$  and  $0 \rightarrow T \rightarrow L^k \rightarrow N \rightarrow 0$ . By the assumption,  $d(K) = d(T) + k$ . Since  $d(K) - k < d(L^k)$ , by the later exact sequence follows  $d(K) \geq d(N) + k + 1$  (resp.  $d(K) = d(N) + k + 1$ ) by  $(\widetilde{d1})$  (resp.  $(d1)$ ).

(2); Let  $k$  be a nonnegative integer. We assume that the assetion is true for  $i = k - 1$  and prove the case of  $i = k$ . There exist the exact sequence  $0 \rightarrow T \rightarrow L^1 \rightarrow L^2 \rightarrow \dots \rightarrow L^k \rightarrow N \rightarrow 0$  and  $0 \rightarrow K \rightarrow L^0 \rightarrow T \rightarrow 0$ . By the assumption,  $d(T) = d(N) + k$ . Since  $d(N) + k < d(L^0)$ , we have  $d(K) = d(N) + k + 1$  by the later exact sequence and  $(d2)$ .  $\square$

We put  $(-)^{\dagger} = \text{Hom}_{\Gamma}(-, C)$ . Let  $\dots \rightarrow P_n \rightarrow P_{n-1} \rightarrow \dots \rightarrow P_1 \xrightarrow{d_1} P_0 \rightarrow M \rightarrow 0$  be the resolution of finitely generated projective modules of  $M$ . Let  $W$  be a  $\Lambda$ -module and  $\lambda_W: W \rightarrow M^{\dagger}$  be a homomorphism of  $\Lambda$ -module. We put

$$E^i(\lambda_W): = \begin{cases} \text{Ker } \lambda_W & (i = 1) \\ \text{Coker } \lambda_W & (i = 2) \\ \text{Ext}_R^{i-2}(M, C) & (i \geq 3) \end{cases}.$$

$$t(\lambda_W): = \inf\{i \geq 0 \mid E^{i+1}(\lambda_W) \neq 0\}.$$

We assume  $\Lambda = \Gamma = R$  is a commutative Noethrian ring,  $M'$  is a finitely generated  $R$ -module,  $C$  is a semidualizing  $R$ -module,  $W = M'$ ,  $M = M'^{\dagger}$  and  $\lambda_{M'}: M' \rightarrow M'^{\dagger}$  is natural map. Then we have  $E^i(\lambda_W) = \text{Ext}_R^i(\text{Tr}_C M', C)$ ; see [3, Proposition 4.13] and see [4] for a detail of a semidualizing module. For  $0 \leq i \leq 2$  we put

$$O^i(\lambda_W): = \begin{cases} \text{Coker}(d_1^{\dagger}: P_0^{\dagger} \rightarrow P_1^{\dagger}) & (i = 0) \\ \text{Im}(d_1^{\dagger}: P_0^{\dagger} \rightarrow P_1^{\dagger}) & (i = 1) \\ W & (i = 2) \end{cases}.$$

Let  $S$  be a non empty set, and  $d_{\square}$  be a function which detemind for  $p \in S$  a function which detemind a nonnegative integer or  $\infty$  for every  $\Lambda$ -modules. Let  $f$  be a function which detemind a nonnegative integer or  $\infty$  for a element of  $S$ . Let  $i$  be an integer and  $U$  be a subset of  $S$ . Let  $N$  be  $\Lambda$ -module. We say that  $N$  satsfies  $\widetilde{S}_i^f$  if  $d_p(N) \geq \inf\{i, f(p)\}$  for an integer  $i$  and all  $p \in S$ . We put

$$\begin{aligned} X_i(N): &= \{p \in S \mid i \leq d_p(N)\}, \\ Y^f(N): &= \{p \in S \mid d_p(N) < f(p)\}, \\ T^U(N): &= \inf\{d_p(N) \mid p \in U\}, \\ \widetilde{S}^f(N): &= \sup\{i \mid N \text{ satisfies } \widetilde{S}_i^f\}. \end{aligned}$$

Note that  $0 \leq T^U(N)$  and  $0 \leq \widetilde{S}^f(N)$ . Let  $i$  be a integer . The following Propositions hold.

**Proposition 2.2.** (1)  $\widetilde{S}^f(N) = \inf\{d_p(M) \mid p \in Y^f(N)\}$  holds.

- (2) *If  $d_p$  satisfies  $(\widetilde{d1})$  (resp.  $(d3)$ ,  $(d1)(d2)(d3)$ ) for any  $p \in S$ , then the function  $T^U$  also satisfies  $(\widetilde{d1})$  (resp.  $(d3)$ ,  $(d1)(d2)(d3)$ ).*

(3) If  $d_p$  satisfies  $(\widetilde{d1})$  (resp.  $(d3)$ ) for any  $p \in S$ , then the function  $\widetilde{S}^f$  also satisfies  $(\widetilde{d1})$  (resp.  $(d3)$ ).

*Proof.* (1); If  $Y^f(N) = \emptyset$ ,  $\widetilde{S}^f(N) = \infty$  follows by definition. Therefore the assertion holds. If  $Y^f(N) \neq \emptyset$ , we put  $l = \min\{d_p(N) \mid p \in Y^f(N)\}$ . We choose  $p \in Y^f(N)$  satisfying  $d_p(N) = l$ . Then, for  $q \notin Y^f(N)$   $d_q(N) \geq \inf\{l, f(p)\}$  holds clearly. By our choice of  $p$  for  $q \in Y^f(N)$ , inequality also holds. Therefore  $l \leq \widetilde{S}^f(N)$ . By the definition of  $\widetilde{S}^f(N)$ ,  $d_p(N) \geq \inf\{\widetilde{S}^f(N), f(p)\}$  holds. Since  $p \in Y^f(N)$ , We get  $\widetilde{S}^f(N) \leq d_p(N) = l$ . Therefore (1) holds.

(2);  $(\widetilde{d1})$ ; Let  $0 \rightarrow K \rightarrow L \rightarrow N \rightarrow 0$  be a short exact sequence of  $\Lambda$ -module. If  $T^U(K) < T^U(L)$ , there exist  $p \in S$  such that  $T^U(K) = d_p(K) < d_p(L)$  and  $p \in U$ . Since  $d_p$  satisfies  $(\widetilde{d1})$ , we have  $d_p(K) \geq d_p(N) + 1$ . Therefore  $T^U(N) + 1 \leq T^U(K)$ .  $(d3)$ ; If  $T^U(L) < T^U(N)$ , there exist  $p \in S$   $T^U(L) = d_p(L) < d_p(N)$ , and  $p \in U$ . Since  $d_p$  satisfies  $(d3)$ ,  $d_p(L) \geq d_p(K)$  Therefore  $T^U(L) \geq T^U(K)$ .  $(d1)(d2)(d3)$ ; We may prove only  $T^U$  satisfies  $(d2)$  by a note below of definition of  $d$ . If we assume  $T^U(N) < T^U(L)$ , there exist  $p \in S$   $T^U(N) = d_p(N) < d_p(L)$  and  $p \in U$ . Since  $d_p$  satisfies  $(d2)$ , we have  $d_p(K) = d_p(N) + 1$ . Therefore  $T^U(N) + 1 \geq T^U(K)$ . If  $T^U(K) = T^U(L)$ , we have also  $T^U(N) + 1 = T^U(K)$  clearly. If  $T^U(K) < T^U(L)$  we have  $T^U(N) + 1 = T^U(K)$  because  $T^U$  satisfies  $(\widetilde{d1})$ . Therefore  $(d2)$  holds.

(3); Let  $0 \rightarrow K \rightarrow L \rightarrow N \rightarrow 0$  be a short exact sequence of  $R$ -module.  $(\widetilde{d1})$ ; If  $\widetilde{S}^f(K) < \widetilde{S}^f(L)$ , there exist  $p \in S$  such that  $\widetilde{S}^f(K) = d_p(K) < f(p)$  by (1), and  $d_p(K) < d_p(L)$ . Since  $d_p$  satisfies  $(\widetilde{d1})$ ,  $d_p(K) \geq d_p(N) + 1$ . Since  $f(p) > d_p(K) > d_p(N)$ , by (1) we have  $\widetilde{S}^f(K) \geq \widetilde{S}^f(N) + 1$ .  $(d3)$ ; We assume  $\widetilde{S}^f(L) < \widetilde{S}^f(N)$ . there exist  $p \in S$  such that  $d_p(L) < d_p(N)$ , and  $d_p(L) < f(p)$ . Since  $d_p$  satisfies  $(d3)$ ,  $d_p(K) \leq d_p(L) < f(p)$ . Therefore  $\widetilde{S}^f(K) \leq \widetilde{S}^f(L)$  by (1).  $\square$

Let  $f$  be the function which is  $f(p) = d_p(C)$  for  $p \in S$ . We put  $\widetilde{S}^C = \widetilde{S}^f$ ,  $Y^C = Y^f$ . The following Proposition give the relationships among  $t$ ,  $T^U$ ,  $\widetilde{S}^C$ ,  $X_i$ ,  $Y^C$ .

**Proposition 2.3.** *We assume that  $d$  satisfies  $(\widetilde{d1})(d3)$ . Let  $N$  be  $\Lambda$ -module and  $C'$  be a direct summand of a finite direct sum of copies of  $C$ . the following holds.*

- (1)  $i \leq T^{X_i(C)}(C')$ ,  $\widetilde{S}^C(C') = \infty$  holds for any integer  $i$ .
- (2) If  $d'$  is a function which satisfies  $(\widetilde{d1})$  and  $t(\lambda_W) + i - 1 \leq d'(C')$  for every  $C'$  a direct summand of a finite direct sum of copies of  $C$ , then  $t(\lambda_W) + i - 2 \leq d'(O^i(\lambda_W))$  holds for  $0 \leq i \leq 2$ .
- (3) We have  $Y^C(O^i(\lambda_W)) \subseteq X_{t(\lambda_W)+i-1}(C)$  for  $0 \leq i \leq 2$ .
- (4)  $t(\lambda_W) + i - 2 \leq T^{X_{t(\lambda_W)+i-1}(C)}(O^i(\lambda_W)) \leq \widetilde{S}^C(O^i(\lambda_W))$  for  $0 \leq i \leq 2$  holds.
- (5)  $t(\lambda_W) + i - 2 = T^{X_{t(\lambda_W)+i-1}(C)}(O^i(\lambda_W))$  if and only if  $t(\lambda_W) + i - 2 = \widetilde{S}^C(O^i(\lambda_W))$  for  $0 \leq i \leq 2$ .

*Proof.* (1);  $i \leq T^{X_i(C)}$ ,  $\widetilde{S}^C(C) = \infty$  is obvious by the definition. There exist a short exact sequence  $0 \rightarrow C \rightarrow \oplus^s C \rightarrow \oplus^{s-1} C \rightarrow 0$  for  $s > 1$ , and there exist  $k > 1$  and a  $\Lambda$ -module  $C''$  such that  $0 \rightarrow C' \rightarrow \oplus^k C \rightarrow C'' \rightarrow 0$  and  $0 \rightarrow C'' \rightarrow \oplus^k C \rightarrow C' \rightarrow 0$  is exact. The assertion follows because  $T^{X_i(C)}$ ,  $\widetilde{S}^C$  satisfies  $(\widetilde{d1})(d3)$  by Proposition 2.2 (2)(3).

(2); The cases of  $t(\lambda_W) = 0$  and  $t(\lambda_W) = 1$ ,  $i = 0, 1$  is clear. If  $t(\lambda_W) = 1$ ,  $i = 2$ , we have  $0 \rightarrow W \rightarrow M^\dagger \rightarrow E^2(\lambda_W) \rightarrow 0$ , and  $0 \rightarrow M^\dagger \rightarrow C^0 \rightarrow C^1 \rightarrow O^0(M) \rightarrow 0$ . Where each  $C^i$  is a direct summand of a finite direct sum of copies of  $C$  for  $0 \leq i \leq 1$ . Since  $d'$  satisfies  $(\widetilde{d1})$ , we have  $d'(M^\dagger) \geq 2$  by (1) and proposition 2.1 (1). Therefore we have  $t(\lambda_W) \leq d'(W)$  by the later exact sequence. We assume that  $t(\lambda_W) \geq 2$ . Then we have  $W \cong M^\dagger$ . Since  $E^i(\lambda_W) = 0$  for  $2 \leq i \leq t(\lambda_W)$ , we have exact sequence  $0 \rightarrow M^\dagger \rightarrow C^0 \rightarrow C^1 \rightarrow \dots \rightarrow C^{t(\lambda_W)-1} \rightarrow L \rightarrow 0$ . Where each  $C^i$  is a direct summand of a finite direct sum of copies of  $C$  for  $0 \leq i \leq t(\lambda_W) - 1$ . Since  $d'$  satisfies  $(\widetilde{d1})$ ,  $t(\lambda_W) + i - 2 \leq d'(O^i(\lambda_W))$  for  $0 \leq i \leq 2$  follows.

(3); Let be  $0 \leq i \leq 2$ . We take  $p \in Y^C(O^i(\lambda_W))$  Then we have  $t(\lambda_W) + i - 2 \leq \widetilde{S}^C(W) \leq d_p(O^i(\lambda_W)) < d_p(C)$  by (2) and Proposition 2.2(1)(3). Therefore  $p \in X_{t(\lambda_W)+i-1}(C)$ .

(4); This is clear by (2)(3) and Proposition 2.2(2)(3).

(5); Let be  $0 \leq i \leq 2$ .  $t(\lambda_W) + i - 2 = T^{X_{t(\lambda_W)+1}(C)}(O^i(\lambda_W))$ , there exist  $p \in S$  such that  $t(\lambda_W) + i - 2 = d_p(O^i(\lambda_W)) < d_p(C)$ . Therefore  $t(\lambda_W) + i - 2 = \widetilde{S}^C(O^i(\lambda_W))$  by (4) and Proposition 2.2 (1). The converse is clear by (4).  $\square$

## 3. THE MAIN RESULT

In this section, we assume  $d_p$  satisfies (d1)(d2)(d3) for any  $p \in S$ . Let  $N$  be  $\Lambda$ -module. We put

$$A_i(N) := \{p \in S \mid d_p(N) = i\}.$$

The following theorem is the main result of this paper which was implied in the introduction.

**Theorem 3.1.** *We assume  $t(\lambda_W) < \infty$ , then*

- (A)  $A_0(E^{t(\lambda_W)+1}(\lambda_W)) \cap X_{t(\lambda_W)+1}(C) = A_{t(\lambda_W)}(W) \cap Y^C(W)$ . Furthermore if  $t(\lambda_W) \geq 2$  we also have  $A_0(E^{t(\lambda_W)+1}(\lambda_W)) \cap X_{t(\lambda_W)+i-1}(C) = A_{t(\lambda_W)+i-2}(O^i(\lambda_W)) \cap Y^C(O^i(\lambda_W))$  for  $0 \leq i \leq 2$ .
- (B)  $A_0(E^{t(\lambda_W)+1}(\lambda_W)) \cap X_{t(\lambda_W)+1}(C) \neq \emptyset$  if and only if  $t(\lambda_W) = \tilde{S}^C(W)$ . Furthermore if  $t(\lambda_W) \geq 2$  we also have  $A_0(E^{t(\lambda_W)+1}(\lambda_W)) \cap X_{t(\lambda_W)+i-1}(C) \neq \emptyset$  if and only if  $t(\lambda_W) + i - 2 = \tilde{S}^C(O^i(\lambda_W))$  for  $0 \leq i \leq 2$ .

*Proof.* (A); If  $t(\lambda_W) = 0$  we have an exact sequence  $0 \rightarrow E^{t(\lambda_W)+1}(\lambda_W) \rightarrow W \rightarrow M^\dagger$ . We take  $p \in A_0(E^{t(\lambda_W)+1}(\lambda_W)) \cap X_{t(\lambda_W)+1}(C)$ . Then we have  $d_p(W) = 0$  by the exact sequence since  $d_p$  satisfies  $(\widetilde{d1})$ . Since  $p \in X_{t(\lambda_W)+1}(C)$  we have  $0 < d_p(C)$ . Therefore we have  $p \in A_{t(\lambda_W)}(W) \cap Y^C(W)$ . Conversely we take  $p \in A_{t(\lambda_W)}(W) \cap Y^C(W)$ . Since  $d_p$  satisfies  $(\widetilde{d1})$ , we get  $d_p(M^\dagger) \geq 1$  by an exact sequence  $0 \rightarrow M^\dagger \rightarrow C^0 \rightarrow O^1(\lambda_W) \rightarrow 0$ . (We always denote a direct summand of a finite direct sum of copies of  $C$  by  $C^i$  for integer  $i$ .) Therefore we have  $d_p(\text{Im}(\lambda_W)) \geq 1$ . Therefore since  $d_p$  satisfies (d3), we get  $d_p(E^{t(\lambda_W)+1}(\lambda_W)) = 0$ . Therefore  $p \in A_0(E^{t(\lambda_W)+1}(\lambda_W)) \cap X_{t(\lambda_W)+1}(C)$ . If  $t(\lambda_W) = 1$ , we have an exact sequence  $0 \rightarrow W \rightarrow M^\dagger \rightarrow E^{t(\lambda_W)+1}(\lambda_W) \rightarrow 0$ . We take  $p \in A_0(E^{t(\lambda_W)+1}(\lambda_W)) \cap X_{t(\lambda_W)+1}(C)$ . Since  $d_p$  satisfies  $(\widetilde{d1})$ , we get  $d_p(M^\dagger) \geq 2$  by an exact sequence  $0 \rightarrow M^\dagger \rightarrow C^0 \rightarrow C^1 \rightarrow O^0(\lambda_W) \rightarrow 0$  and Proposition 2.1 (1). Therefore since  $d_p$  satisfies (d2), we have  $d_p(W) = 1$ . Therefore we have  $p \in A_{t(\lambda_W)}(W) \cap Y^C(W)$ . Conversely we take  $p \in A_{t(\lambda_W)}(W) \cap Y^C(W)$ . Since  $d_p$  satisfies  $(\widetilde{d1})$ , we have  $d_p(E^{t(\lambda_W)+1}(\lambda_W)) = 0$ . Therefore  $p \in A_0(E^{t(\lambda_W)+1}(\lambda_W)) \cap X_{t(\lambda_W)+1}(C)$ . We assume that  $t(\lambda_W) \geq 2$   $0 \leq i \leq 2$ . Then we have  $W \cong M^\dagger$ . There exist the exact sequences  $0 \rightarrow M^\dagger \rightarrow C^0 \rightarrow C^1 \rightarrow \dots \rightarrow C^{t(\lambda_W)-1} \rightarrow L \rightarrow 0$ ,  $0 \rightarrow E^{t(\lambda_W)+1}(\lambda_W) \rightarrow L \rightarrow N \rightarrow 0$ , and  $0 \rightarrow N \rightarrow C^{t(\lambda_W)} \rightarrow U \rightarrow 0$ , where  $L, N, U$  is  $\Lambda$ -modules. We take  $p \in A_0(E^{t(\lambda_W)+1}(\lambda_W)) \cap X_{t(\lambda_W)+i-1}(C)$ . We get  $d_p(L) = 0$  by the second exact sequence because  $d_p$  satisfies  $(\widetilde{d1})$ . We get  $p \in A_{t(\lambda_W)+i-2}(O^i(\lambda_W)) \cap Y_C(O^i(\lambda_W))$  by the first exact sequence and Proposition 2.1 (2) since  $d_p$  satisfies (d2). Conversely we take  $p \in A_{t(\lambda_W)+i-2}(O^i(\lambda_W)) \cap Y^C(O^i(\lambda_W))$ . Then we get  $d_p(L) = 0$  by second exact sequence and Proposition 2.1 (1) since  $d_p$  satisfies (d1). We have  $d_p(N) \geq 1$  by the third exact sequence because  $d_p$  satisfies  $(\widetilde{d1})$ . Therefore since  $d_p$  satisfies (d3), we get  $p \in A_0(E^{t(\lambda_W)+1}(\lambda_W)) \cap X_{t(\lambda_W)+i-2}(C)$ .

(B); We put  $S' = \{1\}$  and  $d'_1 = T^{X_{t(\lambda_W)+1}(C)}$ . (resp. if  $2 \leq t(\lambda_W)$ ,  $d'_1 = T^{X_{t(\lambda_W)+i-1}(C)}$  for  $0 \leq i \leq 2$ ). Then we have  $A_0(E^{t(\lambda_W)+1}(\lambda_W)) \cap X_{t(\lambda_W)+1}(C) = A_{t(\lambda_W)}(W) \cap Y^C(W)$  (resp. if  $t(\lambda_W) \geq 2$   $A_0(E^{t(\lambda_W)+1}(\lambda_W)) \cap X_{t(\lambda_W)+i-1}(C) = A_{t(\lambda_W)+i-2}(O^i(\lambda_W)) \cap Y^C(O^i(\lambda_W))$  for  $0 \leq i \leq 2$ ) for  $S'$  and  $d'_1$  by Proposition 2.2 (2) and (A). We have  $A_0(E^{t(\lambda_W)+1}(\lambda_W)) = A_{t(\lambda_W)}(W)$  (resp. if  $t(\lambda_W) \geq 2$   $A_0(E^{t(\lambda_W)+1}(\lambda_W)) = A_{t(\lambda_W)+i-2}(O^i(\lambda_W))$  for  $0 \leq i \leq 2$ ) by Proposition 2.3 (1). Therefore for  $S$  and  $d$  we get  $A_0(E^{t(\lambda_W)+1}(\lambda_W)) \cap X_{t(\lambda_W)+1}(C) \neq \emptyset$  if and only if  $t(\lambda_W) = T^{X_{t(\lambda_W)+1}(C)}(W)$  (resp. if  $t(\lambda_W) \geq 2$ ,  $A_0(E^{t(\lambda_W)+1}(\lambda_W)) \cap X_{t(\lambda_W)+i-1}(C) \neq \emptyset$  if and only if  $t(\lambda_W) + i - 2 = T^{X_{t(\lambda_W)+i-1}(C)}(O^i(\lambda_W))$  for  $0 \leq i \leq 2$ ). Therefore the assertion follows by Proposition 2.3 (5).  $\square$

We can prove some results of a paper of Dibaei and Sadeghi ([2]) by Theorem 3.1. We recommend to see ([2]) for various notations which was used in their paper and ([4]) for a detail of semidualizing module and  $G_C$ -dimension. We assume that  $\Lambda = \Gamma = R$  is a semiperfect commutative ring and  $C$  is a semidualizing  $R$ -module. We put  $S = \text{Spec } R$ ,  $d_p(N) = \text{depth}_{R_p}(N_p)$  for  $R$ -module  $N$ .

**Corollary 3.2.** [2, Lemma 4.4] *Let  $M$  be a horizontally-linked  $R$ -module of finite and positive  $G_C$ -dimension. Set  $n = \text{r.grade}_R(M, C)$ . If  $\lambda M \in \mathcal{A}_C$  (e.g.  $\text{pd}_R(\lambda M) < \infty$ ), then*

$$\text{Ass Ext}_R^n(M, C) = \{p \in \text{Spec } R \mid G_{C_p}\text{-dim}_{R_p} M_p \neq 0, \text{depth}_{R_p}((\lambda M)_p) = n = \text{r.grade}_{R_p}(M_p, C_p)\}.$$

*Proof.* We put  $W = M^\dagger$ , and  $\lambda_W = id_{M^\dagger}$ . Note we have  $\text{r.grade}_R(M, C) = t(\lambda_W) - 1$ . For  $p \in \text{Ass Ext}_R^n(M, C)$ ,  $G_{C_p}\text{-dim}_{R_p} M_p \neq 0$  and  $n = \text{r.grade}_{R_p}(M_p, C_p)$  is obvious. By  $\lambda M \in \mathcal{A}_C$  we have  $\text{depth}_{R_p}((\text{Tr } M)_p) = \text{depth}_{R_p}((\text{Tr}_C M)_p)$ ; see the proof of [2, Lemma 4.4]. Since  $M$  have positive finite  $G_C$ -dimension we get  $A_0(E^{t(\lambda_W)+1}(\lambda_W)) \subseteq X_{t(\lambda_W)}(C)$  (see [4, Proposition 6.1.7 (vi), 6.4.2]),  $\text{Ass Ext}_R^n(M, C) = \{p \in \text{Spec } R \mid \text{depth}_{R_p}((O^1(\lambda_W))_p) = n < \text{depth}_{R_p}(C_p)\}$  holds by Theorem 3.1. Since  $\text{depth}_{R_p}((O^1(\lambda_W))_p) < \text{depth}_{R_p} C_p$  we have  $\text{depth}_{R_p}((\text{Tr } M)_p) + 1 = \text{depth}_{R_p}((\text{Tr}_C M)_p) + 1 = \text{depth}_{R_p}((O^1(\lambda_W))_p)$  and as  $\text{depth}_{R_p}((\text{Tr } M)_p) < \text{depth}_{R_p} R_p$  we have  $\text{depth}_{R_p}((\text{Tr } M)_p) + 1 = \text{depth}_{R_p}((\lambda M)_p)$ . Therefore  $\text{depth}_{R_p}((O^1(\lambda_W))_p) = \text{depth}_{R_p}((\lambda M)_p)$ . Similarly if  $\text{depth}_{R_p}((\lambda M)_p) < \text{depth}_{R_p}(R_p)$ ,  $\text{depth}_{R_p}((O^1(\lambda_W))_p) = \text{depth}_{R_p}((\lambda M)_p)$  holds. Therefore assertion follows.  $\square$

**Corollary 3.3.** [4, Theorem 4.6 (ii)] *Let  $M$  be a stable  $R$ -module of finite  $G_C$ -dimension and  $\lambda M \in \mathcal{A}_C$  (e.g.  $\text{pd}_R(\lambda M) < \infty$ ). For an integer  $n > 0$ , the following statement hold true.*

*If  $M$  is horizontally linked, then  $\text{r.grade}_R(M, C) \geq n$  if and only if  $\lambda M$  satisfies  $\tilde{S}_n$ .*

*Proof.* We put  $W = M^\dagger$ , and  $\lambda_W = id_{M^\dagger}$ . We may assume  $0 < G_C\text{-dim}(M)$ . Since  $A_0(E^{t(\lambda_W)+1}(\lambda_W)) \subseteq X_{t(\lambda_W)}(C)$ , we have  $t(\lambda_W) - 1 = \tilde{S}^C(O^1(\lambda_W))$  by Theorem 3.1 (B). There exist  $p \in S$  such that  $t(\lambda_W) - 1 = d_p(O^1(\lambda_W)) < d_p(C)$  by Proposition 2.2 (1). Therefore we get  $t(\lambda_W) - 2 = d_p(\text{Tr}_C M)$ . Since  $\lambda M \in \mathcal{A}_C$ , we have  $t(\lambda_W) - 2 = d_p(\text{Tr } M)$ . Therefore we get  $t(\lambda_W) - 1 = d_p(\lambda M) < d_p(R)$ . Therefore we have  $\tilde{S}(\lambda M) \leq t(\lambda_W) - 1$ . Similarly we have also  $\tilde{S}(\lambda M) \geq t(\lambda_W) - 1$ . Therefore the assertion follows.  $\square$

**Corollary 3.4.** [2, Theorem 4.13] *Let  $M$  be a horizontally linked  $R$ -module of finite  $G_C$ -dimension and  $\lambda M \in \mathcal{A}_C$ . then*

$$\text{r.grade}_R(M, C) = \inf\{\text{depth}_{R_p}((\lambda M)_p) \mid p \in NG_C(M)\}$$

*Proof.* We put  $W = M^\dagger$ , and  $\lambda_W = id_{M^\dagger}$ . We may assume  $0 < G_C\text{-dim}(M)$ . Since  $A_0(E^{t(\lambda_W)+1}(\lambda_W)) \subseteq X_{t(\lambda_W)}(C)$ , we have  $A_0(E^{t(\lambda_W)+1}(\lambda_W)) \subseteq X_{t(\lambda_W)-1}(C)$  clearly. Therefore we have  $t(\lambda_W) - 2 = T^{X_{t(\lambda_W)-1}(C)}(\text{Tr}_C M) = T^{Y^C(\text{Tr}_C M)}(\text{Tr}_C M)$  by Theorem 3.1 (B) and Proposition 2.3 (5). Since  $\lambda M \in \mathcal{A}_C$ , we have  $t(\lambda_W) - 2 = T^{X_{t(\lambda_W)-1}(C)}(\text{Tr } M) = T^{Y^C(\text{Tr}_C M)}(\text{Tr } M)$ . Then we get  $t(\lambda_W) - 1 = T^{X_{t(\lambda_W)-1}(C)}(\lambda M) = T^{Y^C(\text{Tr}_C M)}(\lambda M)$  by Proposition 2.2 (2), 2.3 (1) and  $T^{Y^C(\text{Tr}_C M)}(\text{Tr } M) < T^{Y^C(\text{Tr}_C M)}(R)$ . Since  $Y^C(\text{Tr}_C M) \subseteq NG_C(M) \subseteq X_{t(\lambda_W)-1}(C)$ , the assertion follows.  $\square$

**Corollary 3.5.** [2, Theorem 4.12] *Let  $R$  be a local ring,  $M$  an  $R$ -module with  $0 < G_C\text{-dim}_R(M) < \infty$  and  $\lambda M \in \mathcal{A}_C$ . If  $M$  is horizontally linked then the following conditions are equivalent.*

- (i)  $\text{depth}(M) = \text{syz}(M) = \text{r.grade}(\lambda M)$ ;
- (ii)  $m \in \text{Ass}_R(\text{Ext}_R^{\text{r.grade}(\lambda M)}(\lambda M, R))$ ;
- (iii)  $\text{depth } M \leq \text{depth } M_p$  for each  $p \in NG_C(M)$ .

*Proof.* we assume that  $W = M$  and  $\lambda_W; M \rightarrow M^{\dagger\dagger}$  is natural map. Note that  $\text{r.grade}(\lambda M) = \text{r.grade}(\text{Tr } M) - 1$  and (ii) if and only if  $m \in \text{Ass}(\text{Ext}_R^{\text{r.grade}(\text{Tr } M)}(\text{Tr } M, R))$  holds because  $M$  is horizontally linked. We have  $\text{Ext}_R^i(\text{Tr}_C M, C) \cong \text{Ext}_R^i(\text{Tr } M, R)$  for  $i \geq 0$ ; see proof of [2, Theorem 4.6]. We have  $t(\lambda_W) = \tilde{S}^C(M)$  by [2, Proposition 2.4] because  $G_C\text{-dim}_R(M) < \infty$ . Then the assertion follows by Theorem 3.1 (A) and Proposition 2.2(1) since  $NG_C(M) = Y^C(M)$ .  $\square$

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